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On the economic geography of an aging society

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Abstract

In this paper, we study the impacts of the aging of population upon economic geography, constructing a two-region NEG model with overlapping generations. It has been shown that there is a strong tendency for the elderly to agglomerate in one region and that the increase in the proportion of the elderly promotes the spatial agglomeration of economic activities. Furthermore, we show that urban costs somewhat weaken this tendency although the elderly's incentive to agglomerate is still strong.

Keywords: agglomeration; income linkage; location pattern; new economic geography; overlapping generations; surviving rate; urban costs

JEL Classification Numbers: J1; R1; R2

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1 Introduction

One of the most significant changes in demography that many industrialized countries have experienced in recent years is the increase in the elderly population. In Japan, for instance, the proportion of elderly persons 65 years old or older was 12.0% in 1990 but exceeded 20% in 2005 and rose to 26.3% in 2015. We can observe a similar change in a number of countries especially in Asia and Europe, though on a somewhat smaller scale. This change is noteworthy not only in its scale but also in its speed.

It is doubtless that such a significant change affects a spatial distribution of economic activities. A main reason is that the incentives behind a location choice for the elderly differ from those for the young population. In particular, most of the elderly, who do not work, pay little attention to the spatial difference in wage rates while the younger care deeply about such factors. Therefore, it is natural that the regions the elderly choose to live in are different from those the younger do.

Having said that, the impacts of the graying of society on the economic geography are not self-evident, because the decision makings of a younger generation and those of an older generation are linked with each other in at least three respects. First, the primary components of the income for the elderly are savings and pensions, whose amounts are usually independent of their concurrent location, but dependent on their locations in the past periods. Therefore, the decision makings of an older generation have an *income linkage* with those of a younger generation. Second, both generations need to pay urban costs, which are represented by land rents and include commuting costs and the costs associated with negative externalities such as congestion and pollution. The amount of such costs depends on the total population, i.e., the sum of the younger and elderly populations, of a region where they live. In this regard, the location choice of the elderly has a linkage, which we refer to as *urban costs linkage*, with that of the younger. Third and finally, high migration costs attract the elderly toward the region where they used to live, which yields a *migration costs linkage*.

In this paper, we examine how the aging of population affects the economic geography through the changes in the working of agglomeration economies, paying special attention to the difference between the incentives of the elderly and the younger behind location choices. For that purpose, we elaborate a two-region new economic geography (henceforth, NEG) framework with some elements of an overlapping generation model incorporated, which bears the income and the urban costs linkages. To begin with, we examine the basic case without urban costs to obtain following results. First, the elderly are agglomerated in one region while the younger are dispersed over the two regions. The agglomeration of workers in a certain region invokes harsher competition among them, which results in a lower wage in that region, ceteris paribus. To the extent that a working generation seek a higher wage, this works as a centrifugal force. However, it is not relevant to the retired. Second, as we introduce the elderly into the model, the distribution of young workers becomes more biased to the region where the elderly are agglomerated. This is because the agglomeration of the elderly in a region brings about a higher demand for the differentiated product in that region. Third and finally, the rise in a proportion of the elderly promotes the spatial agglomeration of young workers. Next, the model is extended so as to incorporate urban costs. We show that urban costs somewhat weaken the tendency for the elderly to agglomerate in one region although it is still strong.

Despite the significance of the topic, the impacts of the aging of population on economic geography have been seldom studied through a rigorous analysis.[1] One of the most important exception is the work by Gaigné and Thisse (2009). They show that the increase in the retired strengthens the tendency of agglomeration, considering the differences between the incentives of working and retired generations as in this paper. However, they do not take into account the income linkage. Instead, they assume that the income of the elderly consists entirely of the revenue from land rent.[2] Another exception is Naito and Omori (2017), who show, using an overlapping generation model, that agglomeration of workers becomes more likely to occur as the aging advances. However, their model lacks the location choices of the elderly. In other words, their interests are not in the difference between the incentives of old and young populations behind location choices. In addition, the reason for their results is not related to agglomeration economies, but rather to an intrinsic asymmetry between regions.[3] Finally, Sato and Yamamoto (2005) examine the effects of demographic changes upon population concentration. They do not consider, however, the change in the elderly rate.

The rest of the paper consists of four sections. In the next section, we present a basic framework without urban costs. In Section 3, a formal definition of equilibrium is presented. Some properties of equilibrium are derived in Section 4. Section 5 extends the model to include urban costs. Section 6 concludes.

2 Model

Our analysis is based on a new economic geography (henceforth, NEG) framework extended so as to take into account the difference between younger and older consumers' incentives in a location choice. As an NEG framework, we use a two-region footloose entrepreneur model originated

[1]Three reasons are conceivable. First, it may be the case that the change began too recently for researchers to notice its importance. Second, the change is not much noticeable in the United States, where young population have constantly flowed in from abroad. Third and last, the effects are elusive to grasp empirically: too many factors are responsible for the location choices of the elderly in the real economy and controlling them is not straightforward.

[2]This assumption would be questionable in terms of empirical validity. Casual observation tells us that the assumption holds true only for wealthier retirees: average retirees cannot count on rent revenues for their remaining lives but on their own savings and pensions. Furthermore, in their setting, the payments for land rents by young workers are received by the elderly. Thus, an income transfer from a young to an old generation through the rent payments is one of the key elements in their analysis. In contrast, we focus on the income linkage between the two generations.

[3]In their two-region model, capital is used only in an "urban" region, and therefore, the marginal productivity of labor rises as capital increases only in that region while it remains constant in the other "rural" region. When the amount of capital exceeds a critical level, therefore, all the workers decide to move into the urban region, which offers a higher wage.

from Forslid and Ottaviano (2003), which enables us to explicitly solve for an equilibrium. Our model, however, diverges from a conventional model in that the transport cost to ship a differentiated product from one region to the other is assumed so prohibitive that it is indeed geographically *immobile*. This assumption is also adopted in the work by Gaigné and Thisse (2009), among others. Furthermore, we incorporate elements of an overlapping generation model in order to discuss younger consumers' and older consumers' location choices separately. In the research of economic growth, the main arena where the overlapping generation model is used, the capital accumulation arising from consumers' savings occupies a central place in the explanation of the evolution of an economy. In our analysis, however, its role is kept minimal. The purpose is to focus on the role of agglomeration economies.

In an economy, there are two regions (1 and 2); two categories of consumers, which we call worker and entrepreneurs; and two sectors, a constant-returns-to-scale (CRS) sector and a service sector. The former sector produces a homogeneous good, which we call a CRS good, using capital and labor of workers while the latter produces a variety of differentiated "services" using labor of entrepreneurs. The CRS good can be shipped between the two regions with no trade costs. The inter-regional trade costs of services are, in contrast, prohibitive: they cannot be shipped.

Workers and entrepreneurs live for one or two periods. In the first period, they are "young" and work in the CRS sector and the service sector, respectively. At the end of that period, some of them die and the rest survive. The probability that a consumer who was alive at a certain period is still alive at the following period is a given constant and denoted by $s \in (0, 1)$. This survival rate is known to all agents in the economy. If they survive, they become "old" and retire from jobs.

In this paper, we concentrate on a simple case where both the population of young workers and that of young entrepreneurs are invariant over time. In other words, the same masses of young workers and young entrepreneurs are born at each period. We denote these masses by $\bar{L}$ and $\bar{H}$.

Workers cannot migrate between regions: their locations are fixed. To focus on the case symmetric ex ante, we assume that half of them live in region 1 and the other half live in region 2. More specifically, $\bar{L}/2$ young workers and $s\bar{L}/2$ old workers live in each region at every period.

Let us turn to entrepreneurs who work in the service sector. They can freely move between the two regions at the beginning of not only their second period but also their first period. At time t, λ_1^t of young entrepreneurs live in region 1 and λ_2^t of them live in region 2 ($\lambda_1^t + \lambda_2^t = 1$). Out of the surviving entrepreneurs who lived in region 1 at time $t-1$, the portion of μ_1^t choose to remain in region 1 at the beginning of time t while the rest, the portion of $1 - \mu_1^t$, choose to migrate to region 2. μ_2^t denotes the counterpart for the surviving entrepreneurs who lived in region 2 at time $t-1$. Then, the population of old entrepreneurs in region i at time t is equal to $s\bar{H}\left[\mu_i^t \lambda_i^{t-1} + (1 - \mu_j^t)\lambda_j^{t-1}\right]$ ($j \neq i$, $i = 1, 2$). It goes without saying that the total mass of the old entrepreneurs at time t becomes equal to $s\bar{H}$.

We adopt a standard setup of macroeconomics that insurance companies get rid of the uncertainties arising from the risk of death by aggregating individual uncertainties. The companies sell a fair insurance to young consumers with the following clauses: If the consumer who has bought the insurance is alive in the next period, the company pays not only what she has paid but also a dividend (annuity) proportional to her payment. If she is daed, however, all the payment is taken by the insurance company. Let ρ be a dividend rate and i a market interest rate. (For a while, we suppress the superscript for time because it causes no confusion.) An insurance company raises from d dollars of insurance sales a profit equal to $(1+i)d - (1+\rho)sd$. For this insurance to be fair, therefore, ρ must satisfy

$$1+\rho = \frac{1+i}{s}. \tag{1}$$

Because this exceeds $1+i$, consumers indeed buy this insurance. Thanks to this insurance, the assets left by the consumers who die at the end of the first period are redistributed to the surviving consumers.

Workers and entrepreneurs have the same preference over the consumption of a variety of services and the CRS good. Consider a representative consumer who lives in region i when young and lives in region j when old, if she survives. The instantaneous utilities that she obtains from the consumption at the young period and that at the old period are given by

$$u_{yi} = \alpha \ln X_{yi} + (1-\alpha) \ln z_{yi} \qquad \text{and} \qquad u_{oj} = \alpha \ln X_{oj} + (1-\alpha) \ln z_{oj}, \tag{2}$$

respectively. Here, X_{yi} and X_{oj} are aggregate consumptions of services at respective periods:

$$X_{yi} = \left[\int_0^{n_i} x_y(v)^{\frac{\sigma-1}{\sigma}}\, \mathrm{d}v \right]^{\frac{\sigma}{\sigma-1}}, \qquad X_{oj} = \left[\int_0^{n_j} x_o(v)^{\frac{\sigma-1}{\sigma}}\, \mathrm{d}v \right]^{\frac{\sigma}{\sigma-1}},$$

where $x_{yi}(v)$ and $x_{oj}(v)$ are the consumptions of each variant at the two periods, and n_i and n_j are the mass of varieties provided in region i and that provided in region j ($i \in \{1,2\}$ and $j \in \{1,2\}$). Furthermore, z_{yi} and z_{oj} are the amounts of the CRS good at the two periods. α is a share of spending in the services.

The utility over the two periods that the consumer expects at her young period is equal to

$$U_{ij} = u_{yi} + \beta s u_{oj}, \tag{3}$$

where β is a discount factor. Note that this is an *expected* utility, which is the weighted sum of the utility obtained when the consumer survives with probability s and the utility obtained when she dies with probability $1-s$, which is equal to 0.

The budget constraint that a representative consumer with income y faces is given by

$$E_{yi} + \frac{E_{oj}}{1+\rho} = y, \tag{4}$$

where

$$E_{yi} \equiv \int_0^{n_i} p_i(v) x_{yi}(v)\, \mathrm{d}v + p_i^G z_{yi} \qquad \text{and} \qquad E_{oj} \equiv \int_0^{n_j} p_j(v) x_{oj}(v)\, \mathrm{d}v + p_j^G z_{oj}$$

are the instantaneous expenditures at the young and old periods, respectively. $p_i(v)$ is a price of each variant of services produced in region i and p_j^G is a price of the CRS goods in region j.

Consumers make decisions on both locations and consumptions.

At the beginning of their young period, entrepreneurs decide the region to live in during that period: they choose i to maximize U_{ij}. They can freely change their location at the next period, that is, the decision on the location at the young period does not confine the location at the old period. However, their decision at the young period affects the utility at the old period through the amount of wage they earn in the young period. In addition, old entrepreneurs choose their location at the beginning of their second period. That is to say, they choose j so as to maximize u_{oj}.

As to consumptions, both workers and entrepreneurs make a consumption plan over the two periods at the beginning of the first period: They decide $x_{yi}(v)$'s, $x_{oj}(v)$'s, z_{yi} and z_{oj} to maximize (3) subject to (4). Furthermore, surviving consumers can again choose consumption bundles at the beginning of the old period right after they find that they have actually survived. They maximize u_{oj} subject to (4) given the value of E_{yi} that has been already spent. However, a glance at the utility function reveals that their choices are time consistent. In other words, the maximization problem of the instantaneous utility, u_{oj}, at the beginning of the old period gives the same values of $x_{oj}(v)$'s and z_{oj} as the maximization problem of the expected utility over the two periods, U_{ij}, at the beginning of the young period does. It is because the instantaneous utility at each period is independent of the consumptions at the other period given the location decisions, that is, u_{yi} does not depend on X_{oj} nor z_{oj} while u_{oj} does not depend on X_{yi} nor z_{yi}. Consequently, we do not have to consider the old generation's maximization problem.

Now, let us turn our attention to the production side. In the CRS sector, they produce the CRS good through a CRS technology using capital and young workers' labor. The output in each region is given by $K_i^{\theta} L_i^{1-\theta}$ where K_i is the amount of capital used in region i. The CRS good is shipped between the two regions with no transport costs and sold in a competitive market. Therefore, its price becomes equal to each other: $p_1^G = p_2^G \equiv p^G$. Furthermore, capital moves freely between the two regions and therefore, an equal rental price of capital, denoted by r, prevails in the two regions. It follows from these arguments that the wage rates of young workers are equalized in the two regions. We take them as a numeraire. Finally, it follows from profit maximization that

$$K_i = \frac{\theta}{1-\theta} \cdot \frac{\bar{L}}{2r} \quad \text{and} \quad p^G = \frac{1}{1-\theta} \left[\frac{r(1-\theta)}{\theta} \right]^{\theta}. \tag{5}$$

In the service sector, on the contrary, they produce differentiated services through an IRS technology and sell it in a monopolistically competitive market. Each young entrepreneur owns a firm that produces one variety in the region of his/her residence. Thus, the number of varieties produced in region i is equal to the number of young entrepreneurs in that region:

$$n_i = \lambda_i \bar{H} \qquad (i = 1, 2). \tag{6}$$

To produce 1 unit of a variety, each firm uses a units of the CRS good and the skill of its entrepreneur. The revenue that remains after the payment for the input is taken by the entrepreneur, that is, the wage rate of the entrepreneur who produces variety v in region i at period t becomes equal to

$$w_i(v) = \Big[p_i(v) - a\Big]q_i(v), \tag{7}$$

where $q_i(v)$ is the amount of each variant produced in region i. The entrepreneurs sell their varieties at the price that maximizes their own wages.

As has been mentioned in the introduction, we minimize the role of capital accumulation, which occurs as a result of consumers' savings, in order to focus rather on that of agglomeration economies. For that purpose, two ad hoc assumptions are introduced. First, our economy is small and open concerning capital. That is, capital moves freely not only between the two regions there but also all over the world; and thus its rental price is determined in a global market. Given this price, the amount of capital used in the economy is determined through (5) no matter how much capital has been accumulated. The excess of capital flows out abroad while its shortage is supplemented from abroad. Second, capital is owned by those living outside of our economy and therefore, the income of a consumer in our economy consists only of wage.

Finally, because of a technical reason, which will become clear in the following analysis, we limit our attention to the case where σ is greater than 2. This corresponds to the result of empirical studies that σ is indeed much higher than 2 in the real economy.[4]

Assumption 1
$\sigma > 2$.

3 Definition of an equilibrium

In this section, we define an equilibrium.

To begin with, note that maximizing (3) under the constraint of (4) leads us to the well-known fact that the price elasticity of demand for each variety is equal to a constant, $-\sigma$. Therefore, maximizing $w_i(v)$ in (7) implies that

$$p_1(v) = p_2(v) \equiv p \equiv \frac{\sigma}{\sigma - 1}a \quad \text{for any } v, \tag{8}$$

that is, the prices of variants are equal to each other and become a constant. Therefore, (7) implies that

$$w_i(v) = \frac{a}{\sigma - 1}q_i(v) \qquad (i = 1, 2). \tag{9}$$

[4] For instance, Bergstrand et al. (2013) find out that the value is approximately equal to 7. Anderson and Wincoop (2004) review the literature and conclude that the parameter takes 5-10.

Since the prices of all the variants are equal to each other, the demands for each variant are also equal. They are derived from the utility maximization problem as follows:

$$x_{yi}(v) = x_{yi} \equiv \frac{\alpha}{1+\beta s} \cdot \frac{y}{pn_i}, \qquad x_{oj}(v) = x_{oj} \equiv \frac{\alpha\beta s(1+\rho)}{1+\beta s} \cdot \frac{y}{pn_j} \quad \text{for any } v. \tag{10}$$

From now on, we need to distinguish time periods and thus begin to add a superscript if necessary. Aggregating demands for all consumers, we can derive the total demand for each variant:

$$q_i^t(v) = q_i^t \equiv \frac{\alpha}{pn_i^t(1+\beta s)} \left[Y_{yi}^t + \beta s(1+\rho^t) Y_{oi}^t \right] \quad \text{for any } v. \tag{11}$$

Here, Y_{yi}^t is the aggregate income of young consumers in region i at time t, which is given by

$$Y_{yi}^t = \lambda_i^t \bar{H} w_i^t + \frac{\bar{L}}{2}. \tag{12}$$

Note that all the entrepreneurs in the same region earn equal wage because all the variants are sold by the same amount: $w_i^t(v) = w_i^t$ for any v (see (7)). Furthermore, Y_{oi}^t is the aggregate income of old consumers in that region at that period. Because the old consumers at time t earned their incomes at time $t-1$,

$$Y_{oi}^t = s \left[\mu_i^t \lambda_i^{t-1} \bar{H} w_i^{t-1} + (1-\mu_j^t) \lambda_j^{t-1} \bar{H} w_j^{t-1} + \frac{\bar{L}}{2} \right]. \tag{13}$$

The first term inside the square brackets represents the aggregate income of the old entrepreneurs who earned their wage in region i at the precedent period and then remained in that region. The second term represents the counterparts of those who earned their wage in region j and then migrated to region i. Lastly, the last term denotes the income of the old workers in region i.

As has been mentioned, this paper concentrates on the *steady state equilibrium* in which all the variables do not change over time. First, entrepreneurs' decisions on which region to live in are always same: $\mu_i^t = \mu_i$ and $\lambda_i^t = \lambda_i$ for any t ($i = 1,2$). In what follows, we may use slightly different notations, $\lambda \equiv \lambda_1$ along with λ_i's. Second, the wage rates also take constant values: $w_i^t = w_i$ for any t ($i = 1,2$). Third and last, the world capital market is also in a steady state, that is, i and r take the same values over time, respectively. It implies that p^G and K_i are also constants (see (5)) in addition to ρ (see (1)).

It is straightforward to derive the steady-state equilibrium wage rate by solving (8), (9) and (11) along with (12) and (13):

$$w_i = \frac{\alpha(1+\eta)\bar{L}}{2\lambda_i \bar{H}} \cdot \frac{k_j + \alpha\eta(1-\mu_j)}{k_1 k_2 - \alpha^2 \eta^2 \mu_1 \mu_2}, \tag{14}$$

where $\eta \equiv (1+\rho)\beta s^2$ and $k_i \equiv \sigma(1+\beta s) - \alpha(1+\eta\mu_i)$ ($i = 1,2$). Here, we can interpret η as a "social subjective discount factor" in the following sense: If a young consumer postpones her consumption worth 1 dollar until her old period, she will obtain $1+\rho$ dollars when old. However, she discounts the consumption at the old period to a constant fraction of β. In addition, she may die with probability s, which further promotes discounting. Finally, if 1 thousand young consumers postpone their consumption, only s-thousand of those will actually benefit from it. This becomes another force of discounting when we aggregate the benefits over the whole society. Now, note that Assumption 1 implies that $k_1 > 0$, $k_2 > 0$ and $k_1 k_2 - \alpha^2 \eta^2 \mu_1 \mu_2 > 0$.[5] Therefore, w_i's are positive.

To discuss entrepreneurs' decisions on locations, it is necessary to compute indirect utilities. The utility maximization problem implies that the expected indirect utility for the representative entrepreneur with income y, who lives in region i in her young period and region j in her old period, is given by

$$V_{ij}(y) = v_{yi}(y) + \beta s v_{oj}(y), \tag{15}$$

where

$$v_{yi}(y) \equiv \ln y + \frac{\alpha}{\sigma - 1} \ln n_i + k \tag{16}$$

and

$$v_{oj}(y) \equiv \ln y + \frac{\alpha}{\sigma - 1} \ln n_j + k + \ln\Big[(1+\rho)\beta s\Big] \tag{17}$$

are the instantaneous indirect utilities associated with U_y and U_o, respectively, and k is a constant.[6]

When choosing locations, entrepreneurs are myopic in that they care only for instantaneous differences in utility levels, which is a common assumption in the literature.

First of all, the difference for a young entrepreneur is given by $\nu_y \equiv \max\big[V_{11}(w_1), V_{12}(w_1)\big] - \max\big[V_{21}(w_2), V_{22}(w_2)\big]$. Here, the first maximum represents the lifetime indirect utility of an entrepreneur who lives in region 1 when young, while the second maximum represents the counterpart of an entrepreneur who lives in region 2 when young. If $\nu_y > 0$, young entrepreneurs choose to live in region 1 and therefore, $\lambda = 1$. Instead, if $\nu_y < 0$, they prefer living in region 2 and therefore, $\lambda = 0$. If $\nu_y = 0$, they are indifferent and any $\lambda \in [0, 1]$ can be realized.

Next, the difference for an old entrepreneur is equal to $v_{o1}(w_1) - v_{o2}(w_1)$ if she lived in region 1 when young, and to $v_{o1}(w_2) - v_{o2}(w_2)$ if she lived in region 2 when young. However, it turns out that $v_{o1}(y) - v_{o2}(y)$ does not depend on y. Therefore, we can write those differences as $\nu_o \equiv v_{o1}(y) - v_{o2}(y)$ for any y. If $\nu_o > 0$, old entrepreneurs choose to live in region 1 and therefore, $\mu_1 = 1$ and $\mu_2 = 0$. If $\nu_o < 0$, to the contrary, they choose to live in region 2: $\mu_1 = 0$ and $\mu_2 = 1$. Finally, if $\nu_o = 0$, they are indifferent between the two regions and therefore, μ_1 and μ_2 can take any values in $[0, 1]$. However, we limit ourselves to the special case with $\mu_1 = 1 - \mu_2$, where the share of the old entrepreneurs who took up residence in region 1 and remain in that region is equal to the share of those who settled down in region 2 and migrate to region 1. In other words, the share of the old entrepreneurs who live in region 1 is the same no matter where they lived when young. Probably it is because each region has particular intrinsic "attractiveness" that consumers do not care about when $\nu_o \neq 0$. Focusing on this special case enables us to consider

[5] First, $k_i > \sigma(1 + \beta s) - \alpha(1 + \eta)$ since $\mu_i \in [0, 1]$ $(i = 1, 2)$. Since $\alpha(1 + \eta) < 2$, however, $\sigma(1 + \beta s) - \alpha(1 + \eta) > 0$ if $\sigma > 2$. Second, $k_1 k_2 - \alpha^2 \eta^2 \mu_1 \mu_2 = [\sigma(1 + \beta s) - \alpha(1 + \eta\mu_1)] \cdot [\sigma(1 + \beta s) - \alpha(1 + \eta\mu_2)]$. However, $\sigma(1 + \beta s) - \alpha(1 + \eta\mu_i) > 0$ since $\alpha(1 + \eta\mu_i) < 2$ $(i = 1, 2)$.

[6] $k \equiv \alpha \ln(\alpha/p) + (1 - \alpha) \ln\left[(1 - \alpha)/p^G\right] - \ln(1 + \beta s)$.

that $\mu_1 + \mu_2$ is always equal to 1 since $\mu_1 = 1$ and $\mu_2 = 0$ if $\nu_o > 0$, and $\mu_1 = 0$ and $\mu_2 = 1$ if $\nu_o < 0$. Thus, we can remove one variable and denote $\mu_1 \equiv \mu$ and $\mu_2 \equiv 1 - \mu$.

Let us denote the location pattern by $\ell \equiv (\lambda, \mu)$. We can use (6), (14), (16) and (17) to obtain the utility differences. First, tedious computations yield

$$\begin{aligned} \nu_y(\ell) &= (1+\beta s) \ln\left(\frac{w_1}{w_2}\right) + \frac{\alpha}{\sigma - 1} \ln\left(\frac{n_1}{n_2}\right) \\ &= -\frac{(\sigma-1)(1+\beta s) - \alpha}{\sigma - 1} \ln\left(\frac{\lambda}{1-\lambda}\right) + (1+\beta s) \ln \omega(\mu), \end{aligned} \tag{18}$$

where

$$\omega(\mu) \equiv \frac{\sigma(1+\beta s) - \alpha(1+\eta-2\eta\mu)}{\sigma(1+\beta s) - \alpha(1-\eta+2\eta\mu)} \tag{19}$$

is a ratio of the total wage bill paid to the young entrepreneurs in region 1 to the bill paid to those in region 2, that is, $\omega(\mu) \equiv w_1\lambda_1 / w_2\lambda_2$. Note that Assumption 1 implies that both the numerator and the denominator of (19) are positive. Consequently, $\omega(\mu) > 0$. For the following analysis, in addition, it is useful to introduce a new notation for the product of the numerator and denominator of (19):

$$A(\mu) \equiv \Big[\sigma(1+\beta s) - \alpha(1+\eta-2\eta\mu)\Big]\Big[\sigma(1+\beta s) - \alpha(1-\eta+2\eta\mu)\Big] > 0.$$

Second,

$$\nu_o(\ell) = \frac{\alpha}{\sigma-1} \ln\left(\frac{n_1}{n_2}\right) = \frac{\alpha}{\sigma-1} \ln \frac{\lambda}{1-\lambda}, \tag{20}$$

which shows that old consumers are interested only in the relative availability of varieties in the two regions.

Then, the equilibrium is formally defined as follows:

Definition 1 (equilibirum)
A location pattern ℓ is an equilibrium if it satisfies both the following conditions:

$$\begin{cases} \lambda = 1 & \text{if} \quad \nu_y(\ell) > 0 \\ \lambda \in [0,1] & \text{if} \quad \nu_y(\ell) = 0 \\ \lambda = 0 & \text{if} \quad \nu_y(\ell) < 0 \end{cases} \quad \text{and} \quad \begin{cases} \mu = 1 & \text{if} \quad \nu_o(\ell) > 0 \\ \mu \in [0,1] & \text{if} \quad \nu_o(\ell) = 0 \\ \mu = 0 & \text{if} \quad \nu_o(\ell) < 0. \end{cases} \tag{21}$$

One of the simplest ways to discuss the stability of an equilibrium is to introduce an ad hoc dynamic process. Our scenario goes as follows. Entrepreneurs simultaneously determine the region to live as soon as they become old. If they are rational, the equilibrium defined above realizes. Now, suppose that one of entrepreneurs mistakenly chooses to live in the region that is not prescribed by the equilibrium. Then, we think that other entrepreneurs *gradually* migrate across regions in response to the mistake. They migrate from the region where they receive a lower level of indirect utility to the region where they receive a higher level, whenever there is a difference between those levels. Their migration behavior is thus described by the following dynamics:

$$\begin{cases} \dot{\lambda} = b_\lambda \nu_y(\ell) \\ \dot{\mu} = b_\mu \nu_o(\ell) \end{cases} \tag{22}$$

for some $b_\lambda > 0$ and $b_\mu > 0$, which are the adjustment speeds. It is obvious that the equilibrium defined above is indeed a stationary point of this dynamical system. We consider the equilibrium stable if the stationary point is locally asymptotically stable.

4 Properties of the equilibrium

In this section, we derive the equilibrium defined in the previous section and examine its properties.

4.1 Benchmark case with only one generation

Before proceeding to the analysis of a general case, it is worth examining a benchmark case with only young generation existing. Suppose that $s = 0$ and eliminate variable μ. Then, the utility difference for the young generation, (18), is degenerated to

$$\nu_y(\ell) = -\frac{\sigma - 1 - \alpha}{\sigma - 1} \ln\left(\frac{\lambda}{1 - \lambda}\right). \tag{23}$$

It is obvious that there is a unique equilibrium, for which $\lambda = 1/2$. Because $\nu_y(\ell) > 0$ for $\lambda < 1/2$ and $\nu_y(\ell) < 0$ for $\lambda > 1/2$, the equilibrium is asymptotically stable (see (22)).

Proposition 1

In the benchmark case with only young generation ($s = 0$), there exists a unique stable equilibrium, where young entrepreneurs are equally split between the two regions.

4.2 General case with two generations

Now, let us begin the analysis of the general case with two generations. First of all, note that Assumption 1 implies that

$$(\sigma - 1)(1 + \beta s) - \alpha > 0. \tag{24}$$

Therefore, $\nu_y(\ell)$ goes to negative infinity as λ approaches 1. Consequently, there is no possibility that $\nu_y(\ell) > 0$ at the equilibrium (see the first line in the left half part of (21)). Moreover, $\nu_y(\ell)$ goes to positive infinity as λ approaches 0. Therefore, we can conclude that there is no possibility that $\nu_y(\ell) < 0$, either. Hence, the only possibility left is that $\nu_y(\ell) = 0$ holds at the equilibrium. Let us denote the value of λ that solves $\nu_y(\ell) = 0$ as a function of μ by $\lambda^o(\mu)$. Then, the equilibrium must satisfy $\lambda = \lambda^o(\mu)$.

It is convenient to distinguish three cases, the case with $\lambda > 1/2$, that with $\lambda = 1/2$ and that with $\lambda < 1/2$.

For the first case with $\lambda > 1/2$, $\nu_o(\ell) > 0$ (see (20)): old consumers prefer living in the region where more varieties are provided. (21) implies that $\mu = 1$: old entrepreneurs are concentrated in region 1. Thus, provided that $\lambda > 1/2$, the only candidate for an equilibrium is $\ell = \big(\lambda^o(1), 1\big)$. To relieve the burden of notation, let us express the variables evaluated at $\mu = 1$ with a circumflex: $\hat{\ell}(\lambda) \equiv \ell(\lambda, 1)$, $\hat{\nu}_y(\lambda) \equiv \nu_y(\lambda, 1)$, $\hat{\omega} \equiv \omega(1)$ and $\hat{\lambda}^o \equiv \lambda^o(1)$.

We examine the existence and uniqueness of $\hat{\lambda}^o$. First, note that $\hat{\nu}_y(1/2) = (1 + \beta s) \ln \hat{\omega}$, where

$$\hat{\omega} = \frac{\sigma(1+\beta s) - \alpha(1-\eta)}{\sigma(1+\beta s) - \alpha(1+\eta)} > 1. \tag{25}$$

Therefore, $\hat{\nu}_y(1/2) > 0$. Second, we have already seen that $\nu_y(\ell)$ goes to negative infinity as λ approaches 1. Third, (24) implies that $\hat{\nu}_y(\lambda)$ is a decreasing function because

$$\frac{\mathrm{d}\hat{\nu}_y(\lambda)}{\mathrm{d}\lambda} = -\frac{(\sigma-1)(1+\beta s) - \alpha}{(\sigma-1)\lambda(1-\lambda)} < 0. \tag{26}$$

These three observations imply that $\hat{\nu}_y(\lambda)$ cuts the λ-axis once and only once in interval $(1/2, 1)$. In other words, $\hat{\nu}_y(\lambda) = 0$ has a unique solution in that interval, that is, there exists unique $\hat{\lambda}^o \in (1/2, 1)$.

So far, we have shown that there exists a unique equilibrium given by $(\hat{\lambda}^o, 1)$, provided that $\lambda > 1/2$. The next task is to examine its stability. First, note that $\hat{\nu}_y(\lambda) \left\{ \begin{smallmatrix} \geq \\ = \\ < \end{smallmatrix} \right\} 0$ if $\lambda \left\{ \begin{smallmatrix} \leq \\ = \\ > \end{smallmatrix} \right\} \hat{\lambda}^o$. Therefore, $\dot{\lambda} \left\{ \begin{smallmatrix} \geq \\ = \\ < \end{smallmatrix} \right\} 0$ if $\lambda \left\{ \begin{smallmatrix} \leq \\ = \\ > \end{smallmatrix} \right\} \hat{\lambda}^o$. Second, $\hat{\nu}_o(\lambda) > 0$ if $\lambda > 1/2$. Since $\hat{\lambda}^o > 1/2$, therefore, $\hat{\nu}_o(\hat{\lambda}^o) > 0$. It follows from (22) that $\dot{\mu} > 0$. Fig.1 shows these two observations in a phase diagram. Note that the locus of $\nu_y(\ell) = 0$ becomes upward sloping.[7] It follows from the diagram that the unique equilibrium is stable.

Next, let us move to the second case with $\lambda = 1/2$. First, (21) implies that $\lambda = 1/2$ can constitute an equilibrium only if $\nu_y(1/2, \mu) = 0$ or equivalently, $\omega(\mu) = 1$. Therefore, μ must be equal to $1/2$. Second, (6) implies that the equal number of varieties are produced in the two regions. Therefore, old consumers receive an equal level of utility whether they live in region 1 or region 2, and consequently, any $\mu \in [0, 1]$ satisfy the conditions of (21). Consequently, there is a unique equilibrium given by $(1/2, 1/2)$. However, it is not stable: it is a saddle point. To see this, we consider the linear approximation of (22) around a stationary point. Let J be the corresponding Jacobian matrix:

$$J \equiv \begin{bmatrix} \dfrac{\partial \nu_y(\ell)}{\partial \lambda} & \dfrac{\partial \nu_y(\ell)}{\partial \mu} \\ \dfrac{\partial \nu_o(\ell)}{\partial \lambda} & \dfrac{\partial \nu_o(\ell)}{\partial \mu} \end{bmatrix}. \tag{27}$$

[7] A tedious computation yields

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = \frac{A(\mu)\big[(\sigma-1)(1+\beta s) - \alpha\big]}{4\alpha\eta\lambda(1-\lambda)(\sigma-1)(1+\beta s)\big[\sigma(1+\beta s) - \alpha\big]} > 0.$$

Since

$$\mathbf{Tr}\, J = -b_\lambda \cdot \frac{(\sigma-1)(1+\beta s)-\alpha}{\lambda(1-\lambda)(\sigma-1)} < 0 \tag{28}$$

and

$$|J| = -\frac{b_\lambda b_\mu}{A(\mu)} \cdot \frac{4\alpha^2\eta(1+\beta s)\left[\sigma(1+\beta s)-\alpha\right]}{\lambda(1-\lambda)(\sigma-1)} < 0, \tag{29}$$

any stationary point is a saddle point. Fig. 2 shows the phase diagram.

Finally, we examine the remaining case with $\lambda < 1/2$. Because everything becomes symmetric to that in the case with $\lambda > 1/2$, it turns out that there is a unique robust equilibrium given by $\ell = (1-\hat{\lambda}^o, 0)$.

Thus, we have established the following proposition.

Proposition 2

There are two stable equilibria, $\ell = (\hat{\lambda}^o, 1)$ and $\ell = (1-\hat{\lambda}^o, 0)$. There exists no other stable equilibrium.

Regarding this proposition, two points are important.

First, the old entrepreneurs are always concentrated in one region although young entrepreneurs are dispersed over the two regions. The reason is that a centrifugal force working in the decision making process of young entrepreneurs is lacking in the counterpart of the elderly. We can explain this as follows. As the number of young entrepreneurs in a region declines, they are faced by less fierce competition among them, which raises their wage. At the same time, the demand for their services decreases and their wage falls. However, the decrease in demand is restricted to some extent because there are a fixed number of immobile workers in that region. Thus, the competition effect dominates the demand effect and as a result, young entrepreneurs are benefited from the decline in their population. This produces a centrifugal force.

Second, because $\hat{\lambda}^0 > 1/2$, the distribution of young entrepreneurs is biased toward the region where the old entrepreneurs are agglomerated. Comparing with the benchmark case, therefore, we can say that introducing the elderly into the model strengthens the tendency for young entrepreneurs to agglomerate. This is because the concentration of old entrepreneurs in a region brings about higher demand for the services in that region.

Now, in order to see how aging affects economic geography, let us examine the effects upon $\hat{\lambda}^o$ of a change in the surviving probability, s, which is equal to the proportion of old consumers in the total population at the steady state. Totally differentiating $\hat{v}_y(\hat{\lambda}^o)$ yields

$$\mathrm{d}v_y(\hat{\lambda}^o) = -\frac{(\sigma-1)(1+\beta s)-\alpha}{\lambda(1-\lambda)(\sigma-1)}\mathrm{d}\lambda + \left[\beta \ln\left(\frac{w_1}{w_2}\right) + (1+\beta s)\frac{w_2}{w_1}\left.\frac{\mathrm{d}w_1/w_2}{\mathrm{d}s}\right|_{\lambda\,=\,\text{const.}}\right]\mathrm{d}s. \tag{30}$$

To begin with, the first term in the right hand side with $\mathrm{d}\lambda$ is negative: as the share of young entrepreneurs in region 1 (λ) increases, more variety is produced in that region but at the same time, the ratio of wage rates (w_1/w_2) decreases because the competition among them becomes severer in region 1. The former effect is dominated by the latter and consequently, the rise in λ reduces $\nu_y(\hat{\lambda}^o)$.

Next, the first term in the square brackets represents a *direct effect*. Note that $\hat{\nu}_y(\hat{\lambda}^o) = 0$ implies that

$$\ln\left(\frac{w_1}{w_2}\right) = -\frac{\alpha}{(\sigma-1)(1+\beta s)}\ln\left(\frac{\hat{\lambda}^o}{1-\hat{\lambda}^o}\right) < 0 \tag{31}$$

for $\hat{\lambda}^o > 1/2$: entrepreneurs' wage rate is lower in the larger region, because a wider variety of services are provided there. Consequently, the direct effect is negative. As consumers come to expect a longer life period, income level becomes more important for them. Therefore, the expected lifetime utility received when they work in a lower-wage region becomes further lower comparing to the expected lifetime utility received when they work in a higher-wage region.

Finally, the last term in the right hand side represents the effect through the change in wage rate ratio when the distribution of entrepreneurs is fixed. It is further decomposed into two sub-effects:

$$\begin{aligned}\left.\frac{\mathrm{d}w_1/w_2}{\mathrm{d}s}\right|_{\lambda\,=\,\text{const.}} &= \left.\frac{\partial w_1/w_2}{\partial s}\right|_{\rho,\lambda\,=\,\text{const.}} + \left.\frac{\partial w_1/w_2}{\partial \rho}\frac{\mathrm{d}\rho}{\mathrm{d}s}\right|_{\lambda\,=\,\text{const.}} \\ &= \frac{2\alpha\eta(1-\lambda)}{s\lambda\left[\sigma(1+\beta s)-\alpha(1+\eta)\right]^2}\cdot\left[\left\{2(\sigma-\alpha)+\sigma\beta s\right\}-\left\{\sigma(1+\beta s)-\alpha\right\}\right]\end{aligned} \tag{32}$$

The first term in the right hand side of the first line, or equivalently, the term in the first pair of braces in the second line captures the effect through the change in a relative wage rate when the dividend rate, ρ, in addition to λ, remains unchanged. This *wage effect* is positive. As the surviving probability rises, the demand for the variety produced in the larger region relatively increases because old consumers are agglomerated in that region. As a result, the wage rate in the larger region rises more compared to that in the smaller region. Instead, the second term in the right hand side of the first line in (32), or equivalently, the term in the second pair of braces, represents a negative *dividend effect*. As the surviving probability rises, the amount of dividend each consumer receives decreases. Because this gives a negative impact on the consumption in an old period more than that in a young period, the producers in the larger region suffer more than those in the smaller region. Therefore, the relative wage rate of the larger region decreases. (32) shows that the positive wage effect dominates the negative dividend effect. Since the direct effect is negative, however, the sign of the term in the square brackets in (30) is ambiguous.

However, it turns out that as s goes to 0, the size of the direct effect, given by the first term in the square brackets in (30), approaches 0. The reason is as follows. As s goes to 0, η also goes to 0 and therefore, $\hat{\omega}$ approaches 1 (see(25). Therefore, $\hat{\nu}_y(\lambda) = 0$ implies that λ converges to $1/2$, and the wage rate ratio, $w_1/w_2 = \hat{\omega}(1-\lambda)/\lambda$, also converges to 1. On the other hand, the size of the sum of the wage effect and the dividend effect, given by (32), approaches a positive constant. Consequently, the term in the square brackets in (30) becomes positive when s is sufficiently small. Then, $\mathrm{d}\nu_y(\hat{\lambda}^o) = 0$ implies that $\mathrm{d}\hat{\lambda}^o/\mathrm{d}s > 0$. Thus, we have established the following result.

Proposition 3

When s is not too large, the share of young entrepreneurs in the larger region further increases as the surviving probability rises.

Thus, the aging of a population tends to strengthen the spatial agglomeration of economic activities.

5 Extension to the case with urban costs

The above result that old entrepreneurs are concentrated in one region follows from the consequence of our model that the elderly prefer living in the larger region however large its population is. In the reality, however, the concentration of population brings about various sorts of negative impacts on consumers: It will boost land rent and thus housing price in the region of concentration, raise commuting costs through the geographical expansion of a city, and give rise to negative externalities such as congestion and pollution, to name a few. We can conjecture that these factors work as a force against the concentration of old entrepreneurs. In this section, we introduce such "urban costs" into the model. A main finding is that the tendency of concentration is weakened but still strong. Because the main structure of the extended model is similar to that of the basic one, we discuss the steady state from the beginning.

The level of the disutility of urban costs for each consumer depends on the population of the region where she lives. Let N_i be the steady state population of region i. It follows from our definitions that

$$N_i(\ell) = \frac{1+s}{2}\bar{L} + (\lambda_i + s\mu_i)\bar{H}. \tag{33}$$

The sub-utility of a young consumer living in region i and that of an old consumer living in region j are now expressed as

$$u_{yi} = \alpha \ln X_{yi} + (1-\alpha)\ln z_{yi} - \gamma \ln c(N_i) \;\text{ and }\; u_{oj} = \alpha \ln X_{oj} + (1-\alpha)\ln z_{oj} - \gamma \ln c(N_j) \tag{34}$$

instead of (2). Here, $c(\cdot)$, which is an increasing function, is a measure of the disutility of urban costs and γ is its relative importance in consumer's preference. Note that we are focusing on the simple case where the functional form of the disutility for an old consumer is the same as that for a young consumer (both being equal to $c(\cdot)$), and furthermore, the weights of the disutility are also equal between the two types of consumers (both being γ) .

Let us obtain the differences between the utility levels when a consumer lives in region 1 and when she lives in region 2. For a young entrepreneur, the difference is given by

$$\nu_y(\ell) = -\frac{(\sigma-1)(1+\beta s)-\alpha}{\sigma-1}\ln\left(\frac{\lambda}{1-\lambda}\right) + (1+\beta s)\ln\omega(\mu) - \gamma\ln\left[\frac{c\big(N_1(\ell)\big)}{c\big(N_2(\ell)\big)}\right]. \tag{35}$$

Variable $\omega(\mu)$ is still given by (19). Note that $\nu_y(\ell)$ approaches positive infinity as λ goes to 0, and negative infinity as it goes to 1. Therefore, it follows from the definition of equilibrium, (21), that neither $\lambda = 0$ nor $\lambda = 1$ can be supported by an equilibrium. Consequently, $\nu_y(\ell) = 0$ at the equilibrium. As in the basic model, we denote λ that solves this equation by $\lambda^o(\mu)$. Because of the limit properties mentioned above, there necessarily exists such $\lambda^o(\mu)$ in interval $(0, 1)$. For an old entrepreneur, furthermore, the difference is equal to

$$\nu_o(\ell) = \frac{\alpha}{\sigma - 1} \ln\left(\frac{\lambda}{1-\lambda}\right) - \gamma \ln\left[\frac{c\big(N_1(\ell)\big)}{c\big(N_2(\ell)\big)}\right]. \tag{36}$$

Thus, the consumer prefers living in a region if the number of variants produced in that region is relatively large and/or the disutility of urban costs to be paid in that region is relatively small.

5.1 Equilibrium with partial agglomeration

First, we examine the equilibrium at which old entrepreneurs are concentrated in one region, say region 1, while young counterpart are dispersed in the two regions. As in the basic model, we denote the variables evaluated at $\mu = 1$ with a circumflex, that is, $\hat{\lambda}^o \equiv \lambda^o(1)$, $\hat{\nu}_o(\lambda) \equiv \nu_o(\lambda, 1)$, $\hat{\omega} \equiv \omega(1)$, which is still given by (25), and $\hat{N}_i(\lambda) \equiv N_i(\lambda, 1)$ for $i = 1, 2$. A sufficient condition for the concentration in region 1 being supported as an equilibrium is that $\hat{\nu}_o(\hat{\lambda}^o) > 0$ whereas a necessary condition is that $\hat{\nu}_o(\hat{\lambda}^o) \geq 0$. Similarly, for the concentration of old consumers in region 2, a sufficient condition is given by $\nu_o\big(\lambda^o(0), 0\big) < 0$ while a necessary condition is given by $\nu_o\big(\lambda^o(0), 0\big) \leq 0$.

Let us define $\Omega(\mu) \equiv \omega(\mu)/\big[1 + \omega(\mu)\big] < 1$, for which $\hat{\Omega} \equiv \Omega(1) = \hat{\omega}/(1 + \hat{\omega})$. Note that $\hat{\Omega} > 1/2$ since $\hat{\omega} > 1$ (see (25)). Furthermore, we define

$$\Gamma(\ell) \equiv \gamma \bar{H} \cdot \left[\frac{c'\big(N_1(\ell)\big)}{c\big(N_1(\ell)\big)} + \frac{c'\big(N_2(\ell)\big)}{c\big(N_2(\ell)\big)}\right] > 0, \tag{37}$$

which measures the average of the growth rates of urban costs when the population of each region increases. In addition, $\hat{\Gamma}(\lambda) \equiv \Gamma(\lambda, 1)$. Then, we can obtain the following result.

Lemma 1

i) If $\hat{\lambda}^o > \hat{\Omega}$, both $(\hat{\lambda}^o, 1)$ and $(1 - \hat{\lambda}^o, 0)$ are stable equilibria.

ii) If $\hat{\lambda}^o < \hat{\Omega}$, there exists no equilibrium with $\mu = 1$ nor equilibrium with $\mu = 0$.

Proof

For a while, we focus on $(\hat{\lambda}^o, 1)$. Note that $\hat{\lambda}^o \left\{ \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} \right\} \hat{\Omega}$ if $\hat{\lambda}^o/(1 - \hat{\lambda}^o) \left\{ \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} \right\} \hat{\omega}$, which holds if

$$\frac{\alpha}{\sigma - 1} \ln\left(\frac{\hat{\lambda}^o}{1 - \hat{\lambda}^o}\right) + \frac{(\sigma - 1)(1 + \beta s) - \alpha}{\sigma - 1} \ln\left(\frac{\hat{\lambda}^o}{1 - \hat{\lambda}^o}\right) - (1 + \beta s) \ln \hat{\omega} \left\{ \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} \right\} 0 \tag{38}$$

since $\hat{\omega} > 0$ by Assumption 1. However, $\hat{\nu}_y(\hat{\lambda}^o) = 0$ implies that the left hand side of (38) is equal to $\hat{\nu}_o(\hat{\lambda}^o)$. If $\hat{\lambda}^o > \hat{\Omega}$, therefore, we have $\hat{\nu}_o(\hat{\lambda}^o) > 0$, which is a sufficient condition. Therefore, $(\hat{\lambda}^o, 1)$ is an equilibrium. Instead, suppose that $\hat{\lambda}^o < \hat{\Omega}$. Then, $\hat{\nu}_o(\hat{\lambda}^o) < 0$, which violates the necessary condition, $\hat{\nu}_o(\hat{\lambda}^o) \geq 0$. Consequently, there is no equilibrium with $\mu = 1$.

Next, the stability results from the following two observations. First, $\hat{v}_y(\lambda)$ is decreasing at $\hat{\lambda}^o$:

$$\frac{\mathrm{d}\hat{v}_y(\hat{\lambda}^o)}{\mathrm{d}\lambda} = -\frac{(\sigma-1)(1+\beta s)-\alpha}{\hat{\lambda}^o(1-\hat{\lambda}^o)(\sigma-1)} - \hat{\Gamma}(\hat{\lambda}^o) < 0. \tag{39}$$

Therefore, for $\lambda \left\{ \begin{smallmatrix} \leq \\ = \\ > \end{smallmatrix} \right\} \hat{\lambda}^o$, sufficiently close to $\hat{\lambda}^o$, we have $\hat{v}_y(\lambda) \left\{ \begin{smallmatrix} \geq \\ = \\ < \end{smallmatrix} \right\} 0$ and therefore, $\dot{\lambda} \left\{ \begin{smallmatrix} \geq \\ = \\ < \end{smallmatrix} \right\} 0$. Second, we have seen that $\hat{v}_o(\hat{\lambda}^o) > 0$, which implies that $\dot{\mu} > 0$.

The similar reasoning applies to $(1-\lambda^o(0), 0)$: Since $\omega(0) \equiv 1/\hat{\omega}$, we have $N_1(\lambda, 0) \equiv \hat{N}_2(1-\lambda)$ and $N_2(\lambda, 0) \equiv \hat{N}_1(1-\lambda)$, which implies that

$$v_y(1-\lambda, 0) \equiv -\hat{v}_y(\lambda). \tag{40}$$

Therefore, $1-\lambda$ solves $v_y(1-\lambda, 0) = 0$ if and only if $\hat{v}_y(\lambda) = 0$, and consequently, $\lambda^o(0) = 1-\hat{\lambda}^o$. In the same manner as for $(\hat{\lambda}^o, 1)$, furthermore, we can show that $\lambda^o(0) \left\{ \begin{smallmatrix} \leq \\ = \\ > \end{smallmatrix} \right\} \Omega(0)$ if $\nu_y(\lambda^o(0), 0) \left\{ \begin{smallmatrix} \leq \\ = \\ > \end{smallmatrix} \right\} 0$. If $\lambda^o(0) < \Omega(0)$, we have $\nu_y(\lambda^o(0), 0) < 0$, which is a sufficient condition for $(\lambda^o(0), 0)$ being an equilibrium. However, $\lambda^o(0) < \Omega(0)$ is equivalent to $\hat{\lambda}^o > \hat{\Omega}$. Instead, if $\lambda^o(0) > \Omega(0)$, or equivalently, $\hat{\lambda}^o < \hat{\Omega}$, then $\nu_y(\lambda^o(0), 0) > 0$; and there is no equilibrium with $\mu = 0$. Finally, to prove the stability of the equilibrium, it suffices to show that $\mathrm{d}\nu_y(\lambda, 0)/\mathrm{d}\lambda < 0$ at $\lambda = 1-\lambda^o(0)$, or equivalently, $\mathrm{d}\nu_y(1-\lambda, 0)/\mathrm{d}\lambda > 0$ at $1-\lambda = \lambda^o(0)$. However, it follows from (40) that this is the case if and only if $\mathrm{d}\left[-\hat{v}_y(\lambda)\right]/\mathrm{d}\lambda > 0$ at $\lambda = 1-\lambda^o(0) = \hat{\lambda}^o$, which indeed holds as (39) shows. ■

To explicate the relative size of $\hat{\lambda}^o$ and $\hat{\Omega}$, which plays a key role in the above result, it is useful to ask what happens if γ changes when $\mu = 1$. For that purpose, we define $\bar{\lambda}$ as the value of λ that equates the populations in the two regions, that is, $\bar{\lambda} \equiv (1-s)/2$. It follows that $\hat{N}_1(\lambda) \left\{ \begin{smallmatrix} \geq \\ = \\ < \end{smallmatrix} \right\} \hat{N}_2(\lambda)$ if and only if $\lambda \left\{ \begin{smallmatrix} \geq \\ = \\ < \end{smallmatrix} \right\} \bar{\lambda}$. One of the important properties is that $\bar{\lambda} < 1/2$ for $s > 0$. Therefore, $\hat{\Omega}$ becomes greater than $\bar{\lambda}$ since $\hat{\Omega} > 1/2$.

It is straightforward to obtain following three observations (see the Proof of Lemma **??** for the detail). First, as γ goes to 0, $\hat{\lambda}^o$ converges to a value that exceeds $\hat{\Omega}$. Second, $\hat{\lambda}^o$ approaches $\bar{\lambda}$ as γ goes to infinity. Third and last, $\hat{\lambda}^o$ decreases with γ as long as $\hat{\lambda}^o > \bar{\lambda}$. These observations altogether imply that there exists a critical value of γ, denoted by γ^S, that determines the relative size of $\hat{\lambda}^o$ and $\hat{\Omega}$:

Lemma 2
There exists γ^S such that $\hat{\lambda}^o \left\{ \begin{smallmatrix} \geq \\ = \\ < \end{smallmatrix} \right\} \hat{\Omega}$ if $\gamma \left\{ \begin{smallmatrix} \leq \\ = \\ > \end{smallmatrix} \right\} \gamma^S$.

Proof
First, as γ converges to 0, $\hat{\lambda}^o$ approaches the solution to

$$-\frac{(\sigma-1)(1+\beta s)-\alpha}{\sigma-1} \ln\left(\frac{\lambda}{1-\lambda}\right) + (1+\beta s)\ln\hat{\omega} = 0.$$

The solution is greater than $1/2$. For such a limit value, furthermore, the left hand side of (38) becomes equal to $\left[\alpha/(\sigma-1)\right]\ln\left[\hat{\lambda}^o/(1-\hat{\lambda}^o)\right]$, which is positive. Therefore, (38) holds with the last inequality sign and consequently, the limit of $\hat{\lambda}^o$ exceeds $\hat{\Omega}$. Second, we examine what happens if γ goes to positive infinity. Suppose that $\hat{\lambda}^o$ approaches a constant a, which is greater than $\bar{\lambda}$. Then, $c\big(\hat{N}_1(\hat{\lambda}^o)\big)/c\big(\hat{N}_2(\hat{\lambda}^o)\big)$ approaches a constant larger than 1. Therefore, $\hat{v}_y(\hat{\lambda}^o) = 0$ requires that the first term in the right hand side of (35) go to positive infinity, which is achieved when $\hat{\lambda}^o$ approaches 0. However, this contradicts $a > \bar{\lambda}$. Instead, suppose that $\hat{\lambda}^o$ approaches a constant b, which is smaller than $\bar{\lambda}$. Because $c\big(\hat{N}_1(\hat{\lambda}^o)\big)/c\big(\hat{N}_2(\hat{\lambda}^o)\big)$ approaches a constant smaller than 1, $\hat{v}_y(\hat{\lambda}^o) = 0$ demands that $\hat{\lambda}^o$ approach 1, which is a contradiction. Consequently, the only possibility left is that $\hat{\lambda}^o$ approaches $\bar{\lambda}$. Third, we have

$$\frac{\mathrm{d}\hat{\lambda}^o}{\mathrm{d}\gamma} = -\frac{\partial \hat{v}_y(\hat{\lambda}^o)/\partial\gamma}{\mathrm{d}\hat{v}_y(\hat{\lambda}^o)/\mathrm{d}\lambda}. \tag{41}$$

This is negative as long as $\hat{\lambda}^o > \bar{\lambda}$ since $\mathrm{d}\hat{v}_y(\hat{\lambda}^o)/\mathrm{d}\lambda < 0$ (see (39)) and $\partial\hat{v}_y(\hat{\lambda}^o)/\partial\gamma < 0$ for $\hat{\lambda}^o > \bar{\lambda}$. The result immediately follows from these three observations and the fact that $\hat{\Omega} > \bar{\lambda}$. ■

Indeed, we can solve for γ^S explicitly. Substituting $\hat{\lambda}^o = \hat{\Omega}$ into $\hat{v}_y(\hat{\lambda}^o) = 0$ yields the following result:

$$\gamma^S = \frac{\alpha}{\sigma - 1} \cdot \frac{\ln\hat{\omega}}{\ln\left[c\left(\hat{N}_1\right)/c\left(\hat{N}_2\right)\right]}. \tag{42}$$

Since ω is the ratio of wage bills in the two regions, the right hand side is a measure of the relative size of economic activities in region 1 discounted by the relative value of urban costs in that region.

Using the result in Lemma 2, we can restate Lemma 1 as follows:

Proposition 4
i) If $\gamma < \gamma^S$, $\ell = (\hat{\lambda}^o, 1)$ and $\ell = (1 - \hat{\lambda}^o, 0)$ are stable equilibria.
ii) If $\gamma > \gamma^S$, there exists no equilibrium with $\mu = 1$ nor equilibrium with $\mu = 0$.

This is explained as follows. When γ is too high, young entrepreneurs care about urban costs to such a great extent that many of them choose to live in region 2 rather than region 1 where old entrepreneurs are concentrated. As a result, the number of varieties produced in region 1 is too small to give old entrepreneurs an incentive to live in that region. As γ declines, urban costs become less important and region 1 becomes more attractive for the young entrepreneurs. This raises their population in that region ($\hat{\lambda}^o$ increases), which enlarges the number of varieties produced there. If this effect is large enough, old entrepreneurs come to choose to live in region 1 and thus, $(\hat{\lambda}^o, 1)$ becomes an equilibrium outcome. Indeed, we have seen that as γ approaches 0, $\hat{\lambda}^o$ converges to a value that exceeds $\hat{\Omega}$. Consequently, $(\hat{\lambda}^o, 1)$ is supported by an equilibrium when γ is sufficiently small. The basic model in the preceding section describes the extreme case with $\gamma = 0$. We can reiterate these arguments for the equilibrium $(1 - \hat{\lambda}^o, 0)$.

The critical value of γ resembles that of "ϕ" or freeness of trade (the inverse measure of transport costs) in the New Economic Geography literature: both values lie on the boundary of the set for which the agglomeration of mobile consumers of a certain type is supported as an equilibrium. Thus, we refer to it as a *sustain point*, the term widely used in the NEG literature, of γ.

In general, furthermore, the effect of a change in the elderly ratio upon the critical value is ambiguous. However, we can show that it is negative for s that is not too large, if function $c(\cdot)$ is convex and its curvature is sufficiently large. That is, $\mathrm{d}\gamma^S/\mathrm{d}s < 0$ for $s < \tilde{s}$ if

$$c''(N) > \frac{[c'(N)]^2}{c(N)} \quad \text{for any } N, \tag{43}$$

where

$$\tilde{s} \equiv \frac{\sigma - (1+i)(\sigma - \alpha)}{\sigma\beta(1+i)}$$

(for the proof, see the Proof of Proposition 5).

Proposition 5

$d\gamma^S/ds < 0$ for $s < \tilde{s}$ if (43) is satisfied.

Proof

Suppose that $\hat{\lambda}^o = \hat{\Omega}$. First, since

$$\begin{cases} \hat{N}_1 = \dfrac{1+s}{2}\bar{L} + \left[\dfrac{\sigma(1+\beta s) - \alpha(1-\eta)}{2\{\sigma(1+\beta s) - \alpha\}} + s\right]\bar{H} \\ \hat{N}_2 = \dfrac{1+s}{2}\bar{L} + \dfrac{\sigma(1+\beta s) - \alpha(1+\eta)}{2\left[\sigma(1+\beta s) - \alpha\right]}\bar{H}, \end{cases} \tag{44}$$

we have $\hat{N}_1 > \hat{N}_2$. However, (43) implies that $c'(N)/c(N)$ increases with N. Therefore, $c'(\hat{N}_1)/c(\hat{N}_1) > c'(\hat{N}_2)/c(\hat{N}_2)$. Second, note that $\partial\hat{N}_1/\partial s = \bar{L}/2 + (1-k^o)\bar{H}$ and $\partial\hat{N}_2/\partial s = \bar{L}/2 + k^o\bar{H}$, where

$$k^o \equiv \frac{\alpha}{2\left[\sigma(1+\beta s) - \alpha\right]^2}\left[\sigma\beta\eta - \frac{\mathrm{d}\eta}{\mathrm{d}s}\left\{\sigma(1+\beta s) - \alpha\right\}\right].$$

It follows from

$$\left[\sigma(1+\beta s) - \alpha\right]^2 - \sigma\beta\alpha\eta = (1+\beta s)\left[\sigma(\sigma - 2\alpha) + \alpha^2\right] + \sigma\beta^2 s^2\left[\sigma - \alpha(1+\rho)\right] > 0$$

and $\mathrm{d}\eta/\mathrm{d}s = (1+i)\beta > 0$ that $k^o < 1/2$. Consequently, $\partial\hat{N}_1/\partial s > \partial\hat{N}_2/\partial s$. These two observations lead to

$$\frac{\partial c(\hat{N}_1)/c(\hat{N}_2)}{\partial s} = \frac{c(\hat{N}_1)}{c(\hat{N}_2)}\left[\frac{c'(\hat{N}_1)}{c(\hat{N}_1)}\frac{\partial\hat{N}_1}{\partial s} - \frac{c'(\hat{N}_2)}{c(\hat{N}_2)}\frac{\partial\hat{N}_2}{\partial s}\right] > 0. \tag{45}$$

Furthermore, note that $\partial\hat{\omega}/\partial s = 2\alpha k'/\left[\sigma(1+\beta s) - \alpha(1+\eta)\right]^2$, where $k' \equiv -\sigma\beta + (\mathrm{d}\eta/\mathrm{d}s)\left[\sigma(1+\beta s) - \alpha\right]$. Since $k' \left\{ \begin{smallmatrix} \leqq \\ > \end{smallmatrix} \right\} 0$ if $s \left\{ \begin{smallmatrix} \leqq \\ > \end{smallmatrix} \right\} \tilde{s}$, we have $\partial\hat{\omega}/\partial s \left\{ \begin{smallmatrix} \leqq \\ > \end{smallmatrix} \right\} 0$ if $s \left\{ \begin{smallmatrix} \leqq \\ > \end{smallmatrix} \right\} \tilde{s}$. Putting these findings altogether, (42) gives the result. ■

This result says that provided that the following two conditions are satisfied, the agglomeration of old entrepreneurs becomes more likely to be supported as a stable equilibrium when the elderly ratio rises. The conditions are that the elderly ratio is not too large and that the urban costs increase rapidly enough with the population of a region.

In addition, it is straightforward to see that $\tilde{s}$ decreases with σ, β and i whereas it increases with α. Therefore, the tendency that aging of society strengthens the likelihood of agglomeration is more pronounced when σ, β and i are lower and α is higher.

5.2 Equilibrium with full dispersion

Now, let us turn to the equilibrium at which old entrepreneurs are not concentrated in one region, that is, both regions have some old entrepreneurs: $\mu \neq 1$ and $\mu \neq 0$. Then, an equilibrium necessitates $\nu_o(\ell) = 0$, or

$$\frac{\alpha}{\sigma - 1} \ln\left(\frac{\lambda}{1-\lambda}\right) - \gamma \ln\left[\frac{c\big(N_1(\ell)\big)}{c\big(N_2(\ell)\big)}\right] = 0. \tag{46}$$

As we have seen earlier, furthermore, $\nu_y(\ell)$ must be equal to 0 at the equilibrium. Combining $\nu_y(\ell) = 0$ and (46) yields

$$\lambda = \Omega(\mu). \tag{47}$$

The equilibrium is the pair that solves (46) and (47) simultaneously. It is obvious that the symmetric distribution, $\ell = (1/2, 1/2)$ is supported as an equilibrium. There may exist another asymmetric equilibrium depending on the functional form of c.

To examine the stability of such an equilibrium, consider again the Jacobin of (27). In the case with urban costs, we have, instead of (29) and (28),

$$\textbf{Tr}\, J = -b_\lambda \left[\frac{(\sigma-1)(1+\beta s) - \alpha}{\lambda(1-\lambda)(\sigma-1)} + \Gamma(\ell)\right] - b_\mu s \Gamma(\ell) < 0 \tag{48}$$

and

$$|J| = \frac{b_\lambda b_\mu (1+\beta s)}{A(\mu)}\Big[-B_1(\ell) + B_2(\ell)\Gamma(\ell)\Big], \tag{49}$$

where

$$B_1(\ell) \equiv \frac{4\alpha^2\eta\,[\sigma(1+\beta s) - \alpha]}{\lambda(1-\lambda)(\sigma-1)} \text{ and } B_2(\ell) \equiv \frac{sA(\mu)}{\lambda(1-\lambda)} + 4\alpha\eta\,[\sigma(1+\beta s) - \alpha]\,.$$

Because the sign of $|J|$ is ambiguous, we can distinguish two cases: A stationary point is stable if $\Gamma(\ell) > B_1(\ell)/B_2(\ell)$ and therefore, $|J| > 0$ at that point. Instead, it is a saddle point if $\Gamma(\ell) < B_1(\ell)/B_2(\ell)$ and therefore, $|J| < 0$. Roughly speaking, the equilibrium is stable when urban costs rises rapidly enough with the increase in population. Unfortunately, these conditions are too complicated for us to conduct comparative statics. In particular, as a result of a change in a parameter, the equilibrium variables that solve (46) and (47) also change, which further affects the conditions. Therefore, we focus on the symmetric equilibrium, because $\ell = (1/2, 1/2)$ is supported as an equilibrium for *any* values of parameters.

For $\ell = (1/2, 1/2)$, $\Gamma(\ell) \left\{ \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} \right\} B_1(\ell)/B_2(\ell)$ if $\gamma \left\{ \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} \right\} \gamma^B$, where

$$\gamma^B \equiv \frac{c(N^o)}{\bar{H}\, c'(N^o)} \cdot \frac{2\alpha^2\eta}{(\sigma-1)\Big[s\big\{\sigma(1+\beta s) - \alpha\big\} + \alpha\eta\Big]}$$

and $N^o \equiv (1+s)(\bar{L}+\bar{H})/2$ is the population of each symmetric region. We have thus established the following result:

Proposition 6
i) If $\gamma > \gamma^B$, $\ell = (1/2, 1/2)$ is a stable equilibrium.
ii) If $\gamma < \gamma^B$, $\ell = (1/2, 1/2)$ is an equilibrium but unstable.

This proposition states that the symmetric equilibrium is stable if the parameter for the importance of urban costs is greater than a critical value. However, it "breaks" if the parameter is lower than the value. We refer to the critical value as a *break point* considering its similarity to the break point of ϕ, freeness of trade, in the NEG literature.

Note that

$$\frac{\mathrm{d}\gamma^B}{\mathrm{d}s} = -\gamma^B \cdot \left[\frac{\bar{L}+\bar{H}}{2c'(N^o)} \left\{ c''(N^o) - \frac{\left[c'(N^o)\right]^2}{c(N^o)} \right\} + \frac{s\sigma\beta}{s\left\{\sigma(1+\beta s)-\alpha\right\}+\alpha\eta} \right]. \tag{50}$$

This implies the following result.

Proposition 7
$d\gamma^B/ds < 0$ if (43) is satisfied.

Therefore, the symmetric equilibrium becomes more likely to be stable as the ratio of elderly rises, as long as the curvature of urban cost function is sufficiently large.

6 Concluding remarks

This paper has studied the impacts of the aging of population upon economic geography, paying special attention to the difference between the incentives of the elderly and the younger behind location choices. For that purpose, we elaborate a two-region NEG model with overlapping generations, which bears the income and the urban costs linkages. It has been shown that there is a strong tendency for the elderly to agglomerate in one region and that the increase in the proportion of the elderly promotes the spatial agglomeration of economic activities. In addition, we have shown that urban costs somewhat weaken this tendency although the elderly still have a strong incentive to agglomerate in one region.

This piece of research, only one of the first steps to attack the topic, has a number of limitations. Among them, the following seem to be important. First, we have assumed that it takes no cost to ship a differentiated product across regions. This assumption makes a dispersion force so strong for mobile workers (young entrepreneurs in our model) as to prevent them from agglomerating in one region at an equilibrium. In order to have both their dispersion and their agglomeration as equilibrium outcomes, the model necessitates a positive transport cost. Second, not only the transport cost but also the migration cost has been abstracted away in our model. It is arguable, however, that in the reality, the latter cost is considerably high especially for the elderly. It would be necessary to incorporate the migration costs linkage into the model.

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